IB Mathematics

Analysis and Approaches

Standard Level

Functional Worktext Designed To Help Students Meet The Assessment Objectives For The IB Examinations

- Promote quick acquisition and mastery of the necessary mathematical concepts and skills
- Strengthen understanding of concepts through detailed worked examples
- Advance application of concepts to typical exam questions
- Build confident critical thinking skills
- Quick answer reference for efficient revision
- Full solutions for effective learning

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PREFACE

IB Mathematics: Analysis and Approaches (Standard Level) is written in line with the latest syllabus. As a functional textbook, it emphasises the quick acquisition and mastery of the necessary mathematical concepts and skills:

Introduction
Definitions, notations and concise explanations ease the students into each topic by providing them with the necessary prerequisites, background knowledge and/or recap.

Examples
Worked examples illustrate how concepts and results covered can be applied to questions. Students are encouraged to attempt the questions from the examples on their own first before taking a look at the solutions to check their approach and the accuracy of their solutions.

Remark / Note
These are put in place to further elaborate upon the results covered, to place emphasis on the details of any of the results that might be easily overlooked, or to highlight some tips to improve on the efficacy of solving problems.

Summary
For each chapter, key concepts are often presented, after the main content, in a tabular form or a list of bullet points to consolidate the concepts learnt.

Practice Questions with Solutions
For every chapter, questions of varying levels of difficulty are provided for the reader to pick up the various concepts through practice, expose the reader to different question types, and to hone the relevant examination skills. Suggested solutions and numerical answers are provided should the reader want a quick check on the methods and accuracy.

The development of new content in each topic strikes a balance between maintaining a reasonable level of rigour while ensuring the necessary results are laid out in a succinct manner. Concepts are developed in a logical manner and linked whenever possible to help students develop a macroscopic view of Mathematics, particularly in the area of Calculus.

While the target readers are students who have completed their Cambridge IGCSE examinations, it is assumed that the readers have only taken IGCSE Mathematics. For learners with Additional Mathematics background, they can either choose to skip the chapters that have been previously covered, or to use them for revision purposes.

It is hoped that the content of the chapters, together with the features mentioned above, would allow students to grasp concepts fast and equip themselves with the necessary skills to cope with the examinations for Mathematics: Analysis and Approaches at the Standard Level.

Mr Lee Jun Cai
AcesMath!

About Aces Aspire

AcesMath! is a tuition centre in Singapore Bukit Timah area specialising in Mathematics for students taking the IB (Higher Level, Standard Level), GCE A-Level (H1, H2 Mathematics), GCE O-Level (Mathematics, Additional Mathematics) and IGCSE (Mathematics, Additional Mathematics). The center has an established track record of value-adding to the students regardless of their previous mathematics interest and standard, producing a steady stream of stellar scores at examinations.

The IB classes are helmed by Mr Lee Jun Cai and Mr Loo Chee Wee who are both NIE trained and have more than 10 000 hours of teaching experience. The team also has the IB certification (Cat 2 and 3) on curriculum (Analysis and Approaches) and Internal Assessments (IA).

The centre offers both group classes and 1-on-1 consultation sessions for those seeking a more targetted setting. Group classes are capped at 9 students to ensure a healthy and effective student-teacher ratio to optimise the students' learning experience. Both face-to-face and online lessons are available to cater to varying needs of the students.

The teachers also have experience with students from a broad international base. Over the years the centre has taught students from Singapore, Brunei, China, Dubai, England, Indonesia, Japan, Malaysia, Myanmar, South Korea, Thailand, and United States.

Hear from former students who have benefitted from our programmes:

"The diversity of question types from various exam boards, compiled by Mr Lee, has been incredibly helpful in my revision. He is very attentive to the students' abilities, and never hesitates to explain concepts in greater detail whenever needed. At times when I struggled with specific questions in my own revision at home, I would message Mr Lee for help and he'd respond quickly and clearly. With the help of Mr Lee and AcesMath!, I managed to achieve a 7 in HL Maths. Thank you so much!" — ***Yvette Wee***

"Mr Lee is a great teacher who is able to effectively convey knowledge and reinforce mathematical concepts in an engaging manner. Taking this class definitely helped me achieve a 7 in IBDP AAHL math." — ***Oliver Xia***

"AcesMath! is ideal for those seeking professional math tuition. This amazing centre had a major impact in boosting my grades. My tuition teacher, Mr Lee, patiently guided me, enabling me to consistently achieve 7s in IB HL math. I highly recommend AcesMath!." — ***Victoria Zhou***

“Mr Loo is an extremely patient and knowledgeable teacher that helped me overcome my struggles that I faced with IB HL Math in school. For my overall HL Math grade in my 1st year of JC, I achieved 4 points, but after going through Mr Loo’s lessons, I eventually managed to achieve 7 points for HL Math in the IBDP November examinations. Despite asking many questions, he would always take the time, sometimes even after class, to explain difficult concepts to me as simply and clearly as possible.” — ***Edward Tan***

“Mr Loo started teaching me IB Math HL since IB1. He provided fruitful guidance for my math IA and has drastically improved my test scores. I can’t thank him enough for all the exam skills and tips he has given me.” — ***Rudy Zeng***

“I would like to express my gratitude to Mr Loo who has guided me through my IB journey. Math HL is a very tough subject, with Mr Loo’s guidance, I have made significant progress and completed my math IA smoothly.” — ***Tixuan Zhang***

CONTENTS

CHAPTER 1

Sequences and Series

Definitions and notations

A **sequence** is a listing of values, represented by $u_1, u_2, u_3, ..., u_n, ...$ where u_n is the n^{th} term in the sequence.

Remark
Sometimes, T_n is used to represent the n^{th} term instead.

If the "formula" for u_n is known, then one can obtain all terms in the sequence.
For instance, if $u_n = n! - 1$, then the first few terms in the sequence are: 0, 1, 5, 23, 119,…

Given a sequence with finitely many terms, the sum of all its terms is called a **series**.
For a sequence $u_1, u_2, u_3, ...$ the **sum to** its **first *n* terms** is given by $S_n = u_1 + u_2 + u_3 + ... + u_n$.
So one has

$$S_1 = u_1$$
$$S_2 = u_1 + u_2$$
$$S_3 = u_1 + u_2 + u_3$$
$$S_4 = u_1 + u_2 + u_3 + u_4$$

Sometimes, instead of defining the sequence itself, the "formula" for sum to first n terms is given. It is thus important to know the following "conversion" result in order to obtain the expression for u_n given S_n:

$$u_n = S_n - S_{n-1} \text{ for } n \geqslant 2 \text{ and } u_1 = S_1$$

Example 1

Given that the sum to first n terms of a sequence, $S_n = 2^n - 1$, find the n^{th} term.

Solution:

For $n \geqslant 2$, $u_n = S_n - S_{n-1}$
$= (2^n - 1) - (2^{n-1} - 1)$
$= 2^{n-1}$

while $u_1 = S_1 = 2^1 - 1 = 1$ which satisfies $u_n = 2^{n-1}$ when $n = 1$.

So $u_n = 2^{n-1}$ for $n \geqslant 1$.

Remark
One should consider both cases, rather than simply using the $u_n = S_n - S_{n-1}$ result for $n = 1$.
This is because S_0 is not defined.

Arithmetic Progression (AP)

A sequence u_1, u_2, u_3, ... is in **arithmetic progression** if the terms are such that $u_n = u_{n-1} + d$ where d is a constant, called the **common difference**. Usually, we denote the first term, u_1, by a.

For example, 8, 5, 2, –1, –4, ... is an arithmetic progression with first term 8 and common difference –3, while 1, 3, 5, 7, 8, 10 is not an arithmetic progression.

Just by simple listing,

n	1	2	3	4	5
u_n	a	$a + d$	$a + 2d$	$a + 3d$	$a + 4d$

Thus, the first important result is

$$n^{\text{th}} \text{ term of AP: } u_n = a + (n-1)d$$

The next important result involves the sum to first n terms of an AP, S_n. This can be easily derived by listing the series in two opposite ways:

$$S_n = a + (a+d) + \ldots + (a + (n-2)d) + (a + (n-1)d)$$
$$S_n = (a + (n-1)d) + (a + (n-2)d) + \ldots + (a+d) + a$$

By adding both series "column by column", one can thus see that $2S_n = n(2a + (n-1)d)$ so

$$\text{Sum to first } n \text{ terms: } S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(u_1 + u_2)$$

Example 2

An arithmetic progression has first term 50 and common difference –3.

(a) Find the 20th term.

(b) Find the sum of the first 10 terms.

Solution:

(a) The 20th term, $u_{20} = 50 + (20-1)(-3)$
$= -7$

(b) Sum of first 10 terms, $S_n = \frac{10}{2}(2(50) + (10-1)(-3))$
$= 365$

How to determine if a sequence is in AP

One first needs the "formula" for u_n. After which, one only needs to show if $u_n - u_{n-1}$ is a constant.

How to determine if a sequence is NOT in AP

One only has to find three consecutive terms whose consecutive differences are not the same.

Note

1. To assert that a sequence is in AP we use two consecutive general terms u_n and u_{n-1}. It is **not sufficient** to use two particular terms (example: u_7 and u_6).
2. To prove a sequence is not in AP we only need three particular terms.

Example 3

For each of the following sequences, the general term is given. Determine if any of them is in arithmetic progression.

(a) $u_n = 7n - 2$,

(b) $u_n = n^2 - 3n$.

Solution:

(a) If $u_n = 7n - 2$, then $u_{n-1} = 7(n-1) - 2 = 7n - 9$,

$u_n - u_{n-1} = 7n - 2 - (7n - 9) = 7$ which is a constant. So the corresponding sequence is in arithmetic progression.

(b) $u_1 = -2$, $u_2 = -2$ and $u_3 = 0$.

Since $u_2 - u_1 = 0 \neq 2 = u_3 - u_2$, the sequence is not in AP.

Geometric Progression (GP)

A sequence $u_1, u_2, u_3, \ldots$ is in **geometric progression** if the terms are such that $u_n = ru_{n-1}$ where r is a constant, called the **common ratio**. Usually, we denote the first term, u_1, by a.

For example, 2, –6, 18, –54, 162, ... is a geometric progression with first term 2 and common ratio –3, while 1, 2, 4, 8, 17, 34 is not a geometric progression.

Just by simple listing,

n	1	2	3	4	5
u_n	a	ar	ar^2	ar^3	ar^4

Thus, the first important result is

$$n^{\text{th}} \text{ term of GP: } u_n = ar^{n-1}$$

The next important result involves the sum to first n terms of an GP, S_n. This can be easily derived by considering the following two cases:

(A) $r = 1$. Then $S_n = \underbrace{a + a + \dots + a}_{n \text{ times}} = na$.

(B) $r \neq 1$. Then

$$S_n = a + ar + \dots + ar^{n-2} + ar^{n-1}$$
$$rS_n = \quad ar + \dots + ar^{n-2} + ar^{n-1} + ar^n$$

One can thus see that $(r - 1)S_n = ar^n - a$, and so

$$\text{Sum to first } n \text{ terms: } S_n = \underbrace{\frac{a(r^n - 1)}{r - 1}}_{(a)} = \underbrace{\frac{a(1 - r^n)}{1 - r}}_{(b)}$$

Remark

1. For the above two results, one needs to know a, n and r.
2. $r \neq 0$.
3. Expression (a) is useful if $|r| > 1$ while Expression (b) is useful if $|r| < 1$.

Example 4

A geometric sequence is given by $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \dots, 128$.

(a) Find the number of terms in the sequence.

(b) Find the sum of all the terms in the sequence.

Solution:

(a) Note that the sequence is in geometric progression with first term $\frac{1}{8}$ and common ratio 2.

Let n be the number of terms in the sequence.

$$\text{Then } u_n = \left(\frac{1}{8}\right)2^{n-1} = 128$$
$$2^{n-4} = 2^7$$
$$n - 4 = 7$$
$$n = 11$$

There are 11 terms.

(b) Sum of first 11 terms, $S_{11} = \dfrac{\frac{1}{8}(2^{11} - 1)}{2 - 1}$

$$= \frac{2047}{8}$$

How to determine if a sequence is in GP

One first needs the "formula" for u_n. After which, one only needs to show if $\frac{u_n}{u_{n-1}}$ is a constant.

How to determine if a sequence is NOT in GP

One only has to find three consecutive terms whose consecutive ratios are not the same.

Note

1. Like in AP, to assert that a sequence is in GP we use two consecutive general terms u_n and u_{n-1}. It is **not sufficient** to use two particular terms (example: u_7 and u_6).
2. To prove a sequence is not in GP we only need three particular terms.

Example 5

For each of the following sequences, the general term is given. Determine if any of them is in geometric progression.

(a) $u_n = 5(6)^n$,

(b) $u_n = n^2 + 3n$.

Solution:

(a) If $u_n = 5(6)^n$, then $u_{n-1} = 5(6)^{n-1}$,

$\frac{u_n}{u_{n-1}} = \frac{5(6)^n}{5(6)^{n-1}} = 6$ which is a constant. So the corresponding sequence is in geometric progression.

(b) $u_1 = 4$, $u_2 = 10$ and $u_3 = 18$.

Since $\frac{u_2}{u_1} = \frac{5}{2} \neq \frac{9}{5} = \frac{u_3}{u_2}$, the sequence is not in GP.

Sum to infinity of a Geometric Series

For a GP with infinitely many terms, one says that its sum to infinity exists (or the series converges) if S_n tends to a finite value as $n \to \infty$.

In the case of a GP, there is only one criteria for the sum to infinity to exist:

common ratio, r must be such that $-1 < r < 1$ or $|r| < 1$

In this case, we have the fifth important result:

Sum to infinity of a GP: $S_\infty = \frac{a}{1-r}$

Example 6

A geometric sequence is given by $3, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{8}, \ldots$

Explain why the corresponding series converges and find the sum to infinity of this series.

Solution:

Since common ratio r is such that $|r| = \left|-\frac{1}{2}\right| < 1$, the series converges.

Sum to infinity, $S_\infty = \dfrac{3}{1-\left(-\frac{1}{2}\right)} = 2$.

Real World Applications

Arithmetic and geometric progressions have various real world applications, such as population modelling, bank loans and interests, athletes' mileage training programmes.

Example 7

Bank X offers 0.5% interest per year for their savings accounts on the last day of each month. Alice started a saving account on 1 Jan 2020 with initial deposit of \$50 000, and she withdraws \$200 from the account on the first day of each subsequent month.

(a) Find the amount of money Alice has in her savings account at the end of the n^{th} month, starting from 1 Jan 2020.

(b) Find the date on which the total amount in her account first exceed \$55 000.

Solution:

(a) The following tabular method might be useful in helping one make sense of how the bank account fluctuates. Notice that the 0.5% interest corresponds to constantly multiplying by 1.005 at the end of each month, and the trick is to simplify the expressions in each cell in terms of powers of 1.005, without evaluating them:

k	Amount at the start of the k^{th} month	Amount at the end of the k^{th} month
1	50 000	50 000 + (0.005)(50 000) = 50 000(1.005)
2	50 000(1.005) – 200	(50 000(1.005) – 200)1.005 $= 50\,000(1.005)^2 - 200(1.005)$
3	$50\,000(1.005)^2 - 200(1.005) - 200$	$(50\,000(1.005)^2 - 200(1.005) - 200)1.005$ $= 50\,000(1.005)^3 - 200(1.005)^2 - 200(1.005)$ $= 50\,000(1.005)^3 - 200(1.005 + 1.005^2)$
…	…	…
n	$50\,000(1.005)^{n-1}$ $- 200(1.005 + 1.005^2 + \ldots + 1.005^{n-2}) - 200$ $= 50\,000(1.005)^{n-1}$ $- 200(1 + 1.005 + 1.005^2 + \ldots + 1.005^{n-2})$	$50\,000(1.005)^n -$ $200(1.005 + 1.005^2 + \ldots + 1.005^{n-1})$

Notice that the n^{th} row is obtained by observing the pattern from the first three rows, and looking at how each of the powers of 1.005 change.

So the amount of money at the end of the n^{th} month is

$$50\,000(1.005)^n - 200(\underbrace{1.005 + 1.005^2 + \ldots + 1.005^{n-1}}_{\text{GP with first term 1.005 and common ratio 1.005, number of terms } = n-1})$$

GP with first term 1.005 and common ratio 1.005, number of terms = $n - 1$

$$= 50\,000(1.005)^n - 200\left(\frac{1.005(1.005^{n-1} - 1)}{1.005 - 1}\right)$$

$$= 50\,000(1.005)^n - 40\,000(1.005)^n + 40\,200$$

$$= 10\,000(1.005)^n + 40\,200$$

(b) (Using a graphing calculator)

n	Amount at the end of the n^{th} month
78	54 955.46
79	55 029.24 > 55 000
80	55 103.39

So account first exceeds \$55 000 when $n = 79$ i.e. 31 July 2026.

Sigma Notations

The summation or sigma notation is a way to write a sum of values in a concise manner:
If a series is of the form $u_m + u_{m+1} + \ldots + u_n$, then one may "condense" the result by:

$$u_m + u_{m+1} + \ldots + u_n = \sum_{r=m}^{n} u_r$$

Remark

1. r needs to be an integer
2. all r satisfying $m \leqslant r \leqslant n$ must have a corresponding term in the expansion

For example, if a series is given by $-1 + 1 + 3 + \ldots + 11$, then one can write each term in the series in the form $2r + 1$, where the first term corresponds to $r = -1$, and the last term corresponds to $r = 5$. Then one can write the above series in sigma notation:

$$-1 + 1 + 3 + \ldots + 11 = \sum_{r=-1}^{5} (2r + 1)$$

Conversely, one also needs to know how to "expand" out a given sigma notation:

$$\sum_{r=2}^{6} (-1)^r (r!) = 2! - 3! + 4! - 5! + 6!$$

Properties of Sigma Notations

1. $\sum_{r=m}^{n} u_r$ is a series involving $n - m + 1$ terms

2. $\sum_{r=m}^{n} (u_r \pm v_r) = \sum_{r=m}^{n} u_r \pm \sum_{r=m}^{n} v_r$

3. $\sum_{r=m}^{n} au_r = a\sum_{r=m}^{n} u_r$ where a is a constant.

4. $\sum_{r=m}^{n} u_r = \sum_{r=1}^{n} u_r - \sum_{r=1}^{m-1} u_r$ if $1 < m \leqslant n$

   More generally, $\sum_{r=m}^{n} u_r = \sum_{r=k}^{n} u_r - \sum_{r=k}^{m-1} u_r$ if $k < m \leqslant n$

5. (Substitution) $\sum_{r=m}^{n} u_r = \sum_{r=m-k}^{n-k} u_{r+k} = \sum_{r=m+k}^{n+k} u_{r-k}$ where $k \in \mathbb{Z}$ is a constant.

Examples of Types of Sigma Notations

1. $\sum_{r=m}^{n} a = \underbrace{a + a + \ldots + a}_{n-m+1 \text{ terms}} = (n - m + 1)a$ if a is a constant.

2. $\sum_{r=1}^{n} r = 1 + 2 + \ldots + n = \frac{n}{2}(1 + n)$

   More generally, $\sum_{r=m}^{n} r = \frac{n-m+1}{2}(m + n)$ (Arithmetic Series)

3. $\sum_{r=1}^{n} a^r = a + a^2 + \ldots + a^n = \frac{a(a^n - 1)}{a - 1}, a \neq 1$

   More generally, $\sum_{r=m}^{n} a^r = \frac{a^m(a^{n-m+1} - 1)}{a - 1}$ (Geometric Series)

4. $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n + 1)(2n + 1)$

5. $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^2(n + 1)^2$

Remark

The results involving $\sum r^2$ and $\sum r^3$ in points 4 and 5 above are not officially in IB syllabus, hence there is no need to memorise them. However, given the results above in a question, one may be expected to apply the properties of sigma notation to evaluate new summations.

Example 8

Evaluate the sum

$$\sum_{r=6}^{\infty} \frac{1}{2^r}.$$

Solution:

$$\sum_{r=6}^{\infty} \frac{1}{2^r} = \underbrace{\frac{1}{2^6} + \frac{1}{2^7} + \frac{1}{2^8} + \ldots}$$

Sum to infinity of geometric series first term $\frac{1}{2^6}$, common ratio $\frac{1}{2}$

$$= \frac{\frac{1}{2^6}}{1 - \frac{1}{2}}$$

$$= \frac{1}{32}$$

Example 9

Evaluate the sum

$$\sum_{r=3}^{32} 5r - 3.$$

Solution:

$$\sum_{r=3}^{32} 5r - 3 = \underbrace{12 + 17 + 22 + \ldots + 157}$$

Arithmetic series first term 12, n^{th} term 157, $n = 32 - 3 + 1 = 30$

$$= \frac{30}{2}(12 + 157)$$

$$= 2535$$

Remark

Summations of the form $\sum ar + b$ (linear in r) are arithmetic series while those of the form $\sum a^r$ (exponential in r) are geometric series. It is a good practice to list out the first few terms to check and to identify any common differences/ratios.

Example 10

Simplify the following in terms of N:

(a) $\sum_{r=1}^{N}(3^{-r} - 5)$

(b) $\sum_{r=1}^{N}(2r + 1)^3$

Solution:

(a) $\sum_{r=1}^{N}(3^{-r} - 5) = \sum_{r=1}^{N}3^{-r} - \sum_{r=1}^{N}5$ (split up the summation)

geometric series
first term 3^{-1}
common ratio 3^{-1}
N terms

$$= \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^N\right)}{1 - \frac{1}{3}} - 5N$$

$$= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^N\right) - 5N$$

(b) $\sum_{r=1}^{N}(2r + 1)^3 = \sum_{r=1}^{N}(8r^3 + 12r^2 + 6r + 1)$

$$= 8\sum_{r=1}^{N}r^3 + 12\sum_{r=1}^{N}r^2 + 6\sum_{r=1}^{N}r + \sum_{r=1}^{N}1$$

$$= 8 \times \frac{1}{4}N^2(N + 1)^2 + 12 \times \frac{1}{6}N(N + 1)(2N + 1) + 6 \times \frac{1}{2}N(N + 1) + N$$

$$= 2N^2(N + 1)^2 + 2N(N + 1)(2N + 1) + 3(N + 1) + N$$

$$= N(2N(N^2 + 2N + 1) + 2(2N^2 + 3N + 1) + 3N + 3 + 1)$$

$$= N(2N^3 + 8N^2 + 11N + 6)$$

Given $\sum_{r=1}^{n} u_r = \text{f}(n)$, one may need to deduce the different variations of the summation. The aim is to rewrite the new given summation in a form as close to the original $\sum_{r=1}^{n} u_r$ as possible.

For instance, it is given that $\sum_{r=2}^{n} \frac{1}{r(r+1)} = \frac{1}{2} - \frac{1}{n+1}$. Below shows the different variations that can be made and how one can find the new summations correspondingly.

	Variation	General	Example
1.	Changes to the last term	$\sum_{r=1}^{s} u_r = \text{f}(s)$	$\sum_{r=2}^{n+2} \frac{1}{r(r+1)} = \frac{1}{2} - \frac{1}{(n+2)+1}$
		$\sum_{r=1}^{\infty} u_r$ (if it exists)	As $n \to \infty$, $\frac{1}{n+1} \to 0$ $\therefore \sum_{r=2}^{\infty} u_r = \frac{1}{2} - 0 = \frac{1}{2}$
2.	Changes to the first term	$\sum_{r=m}^{n} u_r = \text{f}(s)$	$\sum_{r=10}^{n} \frac{1}{r(r+1)} = \sum_{r=2}^{n} \frac{1}{r(r+1)} - \sum_{r=2}^{9} \frac{1}{r(r+1)}$ $= \left(\frac{1}{2} - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{9+1}\right)$
			$\sum_{r=1}^{n} \frac{1}{r(r+1)} = \sum_{r=2}^{n} \frac{1}{r(r+1)} + \frac{1}{1(1+1)}$ $= \left(\frac{1}{2} - \frac{1}{n+1}\right) + \frac{1}{2}$
3.	Changes to the general term (via substitution)	$\sum_{r=1}^{n} u_{r+k}$	$\sum_{r=2}^{n} \frac{1}{(r+4)(r+3)}$ $= \sum_{r-3=2}^{r-3=n} \frac{1}{(r-3+4)(r-3+3)}$ (rep r by $r-3$) $= \sum_{r=5}^{n+3} \frac{1}{r(r+1)}$ $= \sum_{r=2}^{n+3} \frac{1}{r(r+1)} - \sum_{r=2}^{4} \frac{1}{r(r+1)}$ (see point 2) $= \frac{1}{2} - \frac{1}{(n+3)+1} - \left(\frac{1}{2} - \frac{1}{4+1}\right)$ (see point 1)

Summary

1. A **sequence** u_n is a listing of values
2. A **series** S_n is the sum of terms in a sequence
3. To obtain the expression for u_n given S_n:

$$u_n = S_n - S_{n-1} \text{ for } n \geqslant 2 \text{ and } u_1 = S_1$$

	Arithmetic Progression	**Geometric Progression**
Characteristic	Terms differ by a common difference	Terms differ by a common ratio
u_n	$u_1 + (n-1)d$	$u_1 \cdot r^{n-1}$
S_n	$\frac{n}{2}(2u_1 + (n-1)d)$ OR $\frac{n}{2}(u_1 + u_n)$	$\frac{u_1(1-r^n)}{1-r}$ OR $\frac{u_1(r^n-1)}{r-1}$
S_∞	Does not exist	$\frac{u_1}{1-r}$
To Prove	show $u_n - u_{n-1}$ is a constant	show $\frac{u_n}{u_{n-1}}$ is a constant.
To disprove	find three consecutive terms whose consecutive differences are not the same	find three consecutive terms whose consecutive ratios are not the same

Properties of Sigma Notation	• $\sum_{r=m}^{n} u_r$ has $n - m + 1$ terms • $\sum_{r=m}^{n} (u_r \pm v_r) = \sum_{r=m}^{n} u_r \pm \sum_{r=m}^{n} v_r$ • $\sum_{r=m}^{n} au_r = a\sum_{r=m}^{n} u_r$ where a is a constant. • $\sum_{r=m}^{n} u_r = \sum_{r=k}^{n} u_r - \sum_{r=k}^{m-1} u_r$ if $k < m \leqslant n$
Types	• $\sum_{r=m}^{n} a = (n - m + 1)a$ if a is a constant. • $\sum_{r=1}^{n} r = \frac{n}{2}(1+n)$ • $\sum_{r=1}^{n} a^r = \frac{a(a^n - 1)}{a - 1}$

Practice Questions

1. The fifth term in a geometric sequence is 5 and the tenth term is 125.

(a) Find the first term and the common ratio.

(b) The sum of the first n terms exceeds 10 000. Find the least possible value of n.

2. **(a)** **(i)** Find the sum of all integers, between 100 and 3000, which are divisible by 9.

(ii) Express the above sum using sigma notation.

(b) An arithmetic sequence has first term 800 and common difference –3. The sum of the first n terms of this sequence is negative. Find the least value of n.

3. A geometric sequence has a negative common ratio. The sum of the first term and third term is 30. The sum to infinity is 16. Find the common ratio and the first term.

4. Find the sum of all the multiples of 11 between 100 and 600. Hence find the sum of all numbers between 100 and 600 (inclusive of end points) which are not divisible by 11.

5. The sum of the first two terms of an arithmetic sequence is 10, and the sum of the first four terms is 30.

(a) Find the first term;

(b) Find the sum of the first ten terms.

6. The sum of the first n terms of an arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 25n - n^2$. Three terms of this sequence, u_1, u_m and u_{10}, are consecutive terms in a geometric progression. Find m.

7. A circular cake is cut into nineteen slices whose areas are in an arithmetic sequence. The angle of the largest slice is thrice the angle of the smallest slice. Find the common difference and the size of the angle of the smallest slice.

Quick Answer Key

Solutions page

1. **(a)** $r = 5^{\frac{2}{5}}$, $a = 5^{-\frac{3}{5}}$
(b) 16
S1

2. **(a)** **(i)** 499 905
(ii) $\sum_{n=12}^{333} 9n$
(b) 535
S1

3. $-\frac{1}{2}$, 24 S1

4. 15 840, 159 510 S1

5. **(a)** $\frac{15}{4}$
(b) 150
S1

6. 7 or 19 S1

7. $\frac{\pi}{171}$, $\frac{\pi}{19}$ S1

Quick Answer Key

Solutions page

8. The three terms a, 1, b are in geometric progression. The three terms 1, a, b are in arithmetic progression. Find the value of a and of b given that $a \neq b$.

8. $a = -\frac{1}{2}$, $b = -2$ S1

9. The first and fifth terms of a geometric sequence are 10 and $\frac{2}{125}$ respectively.

Given that the terms in the sequence are strictly decreasing, find

(a) the sum of the first n terms of the series;

(b) the sum to infinity of the series.

9. **(a)** $\frac{25}{2}\left(1-\left(\frac{1}{5}\right)^n\right)$

(b) $\frac{25}{2}$ S1

10. The fourteenth, fifth and second terms of an arithmetic sequence form the first three terms of a geometric sequence.

The arithmetic sequence has first term a and non-zero common difference d.

(a) Show that $d = 2a$.

The fifth term of the arithmetic sequence is 3. The sum of the first n terms in the arithmetic sequence exceeds the sum of the first n terms in the geometric sequence by at least 300.

(b) Find the least value of n for which this occurs.

10. **(b)** 31 S2

11. The n^{th} term of a sequence G is given by $u_n = p^{3-n} - p^{2-n}$, where $p \in \mathbb{R}\backslash\{0, 1\}$. It is also known that $(u_2 - u_1)$, $(u_3 - u_2)$ and $(u_1 + u_3)$ are three consecutive terms of an arithmetic progression.

(a) Show that G is a geometric progression.

(b) Find the value of p.

11. **(b)** $\frac{1}{3}$ S2

12. The sixth, ninth and fifteenth terms of a convergent geometric progression G are the first three consecutive terms of an arithmetic progression A.

(a) Show that the common ratio of G is 0.852, correct to 3 decimal places.

Given that the first term of G is 5 and the common ratio of G is the one found above,

(b) evaluate the sum to infinity of G,

(c) find the sum of the first 10 even-numbered terms of A.

12. **(b)** 33.8

(c) Required Sum $= -63.2$ S2

13. An arithmetic progression has first term a and common difference d, where $a > 0$ and the sum of the first five terms is equal to the sum of the first twenty-five terms. Show that $d = -\frac{2}{29}a$. By expressing u_n in terms of a and n or otherwise, find the value of n such that S_n is maximum.

13. 15 S2

14. The seventh term of an arithmetic sequence is equal to 3 and the sum of the first 14 terms is 31.5. Find the first term and the common difference.

14. $a = 12$, $d = -1.5$ S2

15. A geometric sequence has non-zero first term and sum to infinity S. Given that the first, third and fourth terms are consecutive terms of an arithmetic sequence, find the common ratio of the geometric sequence.

If $S = 20$, find the least value of m for which the magnitude of the difference between S and the sum of the first m terms of the geometric sequence is less than 0.005.

15. $r = -0.618$; 18 S2

16. An arithmetic progression has first term a and common difference d, where a and d are non-zero. The first three terms of a geometric progression are equal to the tenth, sixth and third terms respectively of the arithmetic progression.

(a) Prove that the geometric progression is convergent and find, in terms of a, the sum to infinity.

(b) Given that $a > 0$, find the largest value of n for which the sum of the first n terms of the geometric progression is less than four fifths of the sum to infinity.

16. (a) $\frac{64}{7}a$
(b) 5
S3

17. A robot was at the origin (0, 0). It first moves 2 units along the positive x-axis to the point (2, 0). It then turns left and goes up 3 units from its current point, reaching point (2, 3). If the robot continues turning left and going $\frac{3}{2}$ the distance it had previously walked, find the coordinates of the point that the robot will be at after 11 moves.

17. (−79.2, 54.2) S3

Quick Answer Key
Solutions page

18. Due to energy loss to surroundings, a ball at rest from a height of h above horizontal floor will hit the floor and rebound to a height ah, where a is a constant such that $0 < a < 1$. The ball is initially released from a height H above the horizontal floor.

(a) Write down an expression, in terms of H, a and n, for the height to which the ball rises after the n^{th} bounce.

Given that $a = \frac{7}{8}$,

(b) find the number of times the ball has bounced so that the height risen is less than $\frac{H}{30}$ for the first time.

(c) calculate, in terms of H, the total distance covered by the ball just before it comes to a complete rest.

18. (a) $a^n H$
(b) 26
(c) $15H$
S3

19. ECBC Bank charges an interest rate of 3% per month for outstanding balances on its credit cards at the end of the month. If no payment is made by the end of the month, an additional late charge of \$60 is also added on to the outstanding balance. Mr Tan lost his job suddenly, and thus, was unable to make any payments for his credit card bill. On 1 Jan 2018, his outstanding balance on his ECBC credit card was \$2500.

(a) Show that the outstanding balance on Mr Tan's ECBC card will be \$2774.05 on 28 Feb 2018.

(b) Show that the outstanding balance on Mr Tan's ECBC card after n months is $a(1.03^n) + b$, where a and b are constants to be determined.

Mr Tan has another card with BCC Bank with outstanding balance of also \$2500 on 1 Jan 2018. BCC Bank's interest rate per month is 5% but there is no additional late charge for no payment.

(c) Assuming no payment is made for both cards, the outstanding balance on his BCC card would exceed the outstanding balance on his ECBC card after m months. Find the least value of m.

19. (b) $a = 4500$, $b = -2000$
(c) 9
S3

20. Express the following as a series, giving the first and last two terms:

(a) $\sum_{i=1}^{85} i^3$

(b) $\sum_{i=11}^{63} (-1)^i\, i!$

(c) $\sum_{i=-10}^{21} \frac{1}{2i+1}$

21. Express the following series as a sigma notation:

(a) $-1 + 2^2 - 3^2 + \dots + 98^2$

(b) $(1 + x)^n$

(c) $\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{4}} + \frac{3}{\sqrt{6}} - \dots + \frac{51}{\sqrt{102}}$

22. Evaluate the following:

(a) $\sum_{r=-17}^{68} 2$

(b) $\sum_{r=-3}^{30} (2r + 1)$

(c) $\sum_{r=5}^{15} (2^r - 2)$

23. Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, evaluate each of the following:

(a) $\sum_{r=1}^{10} r^2$

(b) $\sum_{r=1}^{N+2} r^2$

(c) $\sum_{r=10}^{20} r^2$

Quick Answer Key

Solutions page

20. (a) $1^3 + 2^3 + \dots + 84^3 + 85^3$

(b) $-11! + 12! - \dots + 62! - 63!$

(c) $-\frac{1}{19} - \frac{1}{17} + \dots + \frac{1}{41} + \frac{1}{43}$

S3

21. (a) $\sum_{i=1}^{98} (-1)^i i^2$

(b) $\sum_{i=0}^{n} \binom{n}{i} x^i$

(c) $\sum_{i=1}^{51} \frac{(-1)^{i+1} i}{\sqrt{2i}}$

S4

22. (a) 172

(b) 952

(c) 65 482

S4

23. (a) 385

(b) $\frac{1}{6}(N+2)(N+3)(2N+5)$

(c) 2585

S4